Final Exam Introduction to Logic 2019/20 (AI and MA + Guests)

Wednesday 29 January, 2020, 08:30–11:30

Model Answers

1: Translating to propositional logic (10 points) Translate the following sentences to *propositional logic*. Atomic sentences are represented by uppercase letters. Provide one translation key for both sentences.

a. They celebrate New Year's Eve only if they invite friends home.

b. They invite friends home unless the dishwasher is broken or they are too tired.

Translation key:
L: they celebrate New Year's Eve
I: they invite friends home
D: the dishwasher is broken
T: they are too tired

Translations:

a. $L \to I$

b. $I \vee (D \vee T)$
Other correct options: $\neg I \to (D \vee T)$ or $\neg(D \vee T) \to I$.

2: Translating to first-order logic (10 points) Translate the following sentences to *first-order logic*. Provide one translation key for both sentences. The domain of discourse is the set of all people.

a. Everyone knows someone who is not the murderer but is afraid of the murderer.

b. Everyone other than Harriet is afraid of the murderer but Harriet is not and if anyone can find the murderer, Harriet can.

Translation key:
h: Harriet
m: the murderer
$A(x,y)$: x is afraid of y
$F(x,y)$: x can find y
$K(x,y)$: x knows y

Translations:
a. $\forall x \exists y (K(x,y) \wedge y \neq m \wedge A(y,m))$
b. $\forall x (x \neq h \to A(x,m)) \wedge \neg A(h,m) \wedge \forall z (F(z,m) \to F(h,m))$
also OK e.g. $\forall x (x = h \leftrightarrow \neg A(x,m)) \wedge (\exists z F(z,m) \to F(h,m))$

3: Formal proofs (20 points) Give formal proofs of the following inferences. In items c and d, P, Q and R are predicate symbols. Do not forget the justifications. Only use the Introduction and Elimination rules and the Reiteration rule.

a.
- Premise: $A \lor (A \land B)$
- Conclusion: $A \land (A \lor B)$

b.
- (no premises)
- Conclusion: $\neg(P \to Q) \to (P \land \neg Q)$

c.
- Premises: $c = a \;\lor\; b = c$, $P(a) \land P(b)$
- Conclusion: $\exists x(P(x) \land x = c)$

d.
- Premise: $\exists y \forall x(P(x) \to R(x, y))$
- Conclusion: $\forall x \exists y(P(x) \to R(x, y))$

a.

1.	$A \lor (A \land B)$	
2.	$\mid A$	
3.	$\mid A$	Reit: 2
4.	$\mid A \land B$	
5.	$\mid A$	$\land$ Elim: 4
6.	A	$\lor$ Elim: 1, 2–3, 4–5
7.	$A \lor B$	$\lor$ Intro: 6
8.	$A \land (A \lor B)$	$\land$ Intro: 6, 7

b.

1.	$\mid \neg(P \to Q)$	
2.	$\mid \mid \neg(P \land \neg Q)$	
3.	$\mid \mid \mid P$	
4.	$\mid \mid \mid \mid \neg Q$	
5.	$\mid \mid \mid \mid P \land \neg Q$	$\land$ Intro: 3,4
6.	$\mid \mid \mid \mid \bot$	$\bot$ Intro: 5, 2
7.	$\mid \mid \mid \neg\neg Q$	$\neg$ Intro: 4–6
8.	$\mid \mid \mid Q$	$\neg$ Elim: 7
9.	$\mid \mid P \to Q$	$\to$ Intro: 3–8
10.	$\mid \mid \bot$	$\bot$ Intro: 9, 1
11.	$\mid \neg\neg(P \land \neg Q)$	$\neg$ Intro: 2–10
12.	$\mid P \land \neg Q$	$\neg$ Elim: 11
13.	$\neg(P \to Q) \to (P \land \neg Q)$	$\to$ Intro: 1–12

c.

1. $c = a \ \lor \ b = c$	
2. $P(a) \land P(b)$	
3. $c = a$	
4. $c = c$	$=$ Intro
5. $a = c$	$=$ Elim: 4, 3
6. $P(a)$	$\land$ Elim: 2
7. $P(a) \land a = c$	$\land$ Intro: 6, 5
8. $\exists x(P(x) \land x = c)$	$\exists$ Intro: 7
9. $b = c$	
10. $P(b)$	$\land$ Elim: 2
11. $P(b) \land b = c$	$\land$ Intro: 10, 9
12. $\exists x(P(x) \land x = c)$	$\exists$ Intro: 11
13. $\exists x(P(x) \land x = c)$	$\lor$ Elim: 1, 3–8, 9–12

d.

1. $\exists y \forall x(P(x) \to R(x, y))$	
2. [a]	
3. [b] $\forall x(P(x) \to R(x, b))$	
4. $P(a) \to R(a, b)$	$\forall$ Elim: 3
5. $\exists y(P(a) \to R(a, y))$	$\exists$ Intro: 4
6. $\exists y(P(a) \to R(a, y))$	$\exists$ Elim: 1, 3–5
7. $\forall x \exists y(P(x) \to R(x, y))$	$\forall$ Intro: 2–6

4: Truth tables (10 points) Answer the following questions using truth tables. Write down the complete truth tables and motivate your answers. Order the rows in the truth tables as follows:

A	B	C	...
T	T	T	...
T	T	F	...
T	F	T	...
T	F	F	...
F	T	T	...
F	T	F	...
F	F	T	...
F	F	F	...

RightOf(a, b)	$b = c$	RightOf(a, c)	...
T	T	T	...
T	T	F	...
T	F	T	...
T	F	F	...
F	T	T	...
F	T	F	...
F	F	T	...
F	F	F	...

a. Check with a truth table whether the following formula is a tautology.

$$((A \to (B \lor C)) \lor (\neg A \to (\neg B \lor C)))$$

b. Check with a truth table whether the following formula is Tarski's World-(TW-)possible. Indicate clearly which rows are spurious (if any).

$$(\neg \mathsf{RightOf}(a, b) \to b = c) \land (\mathsf{RightOf}(a, c) \to \neg(b = c))$$

a.

A	B	C	$(A$	$\to$	$(B$	$\lor$	$C))$	$\lor$	$(\neg$	A	$\to$	$(\neg$	B	$\lor$	$C))$
T	T	T	T	T	T	T	T	**T**	F	T	T	F	T	T	T
T	T	F	T	T	T	T	F	**T**	F	T	T	F	T	F	F
T	F	T	T	T	F	T	T	**T**	F	T	T	T	F	T	T
T	F	F	T	F	F	F	F	**T**	F	T	T	T	F	T	F
F	T	T	F	T	T	T	T	**T**	T	F	T	F	T	T	T
F	T	F	F	T	T	T	F	**T**	T	F	F	F	T	F	F
F	F	T	F	T	F	T	T	**T**	T	F	T	T	F	T	T
F	F	F	F	T	F	F	F	**T**	T	F	T	T	F	T	F

In all rows there is a '**T**' under the main connective. Hence this formula *is a tautology.*

b.

	spurious	RightOf(a,b)	$b=c$	RightOf(a,c)	$(\neg$	RightOf(a,b)	$\to$	$b=c)$	$\land$	(RightOf(a,c)	$\to$	$\neg$	$(b=c))$
1		T	T	T	F	T	T	T	**F**	T	F	F	T
2	*	T	T	F	F	T	T	T	**T**	F	T	F	T
3		T	F	T	F	T	T	F	**T**	T	T	T	F
4		T	F	F	F	T	T	F	**T**	F	T	T	F
5	*	F	T	T	T	F	T	T	**F**	T	F	F	T
6		F	T	F	T	F	T	T	**T**	F	T	F	T
7		F	F	T	T	F	F	F	**F**	T	T	T	F
8		F	F	F	T	F	F	F	**F**	F	T	T	F

There are non-spurious rows with a '**T**' under the main connective, namely rows 3, 4 and 6. Hence this formula *is TW-possible.*

5: Normal forms propositional logic (5 points) Provide a conjunctive normal form (CNF) of the following formula. Show all of the intermediate steps. You do not have to give justifications.

$$(B \to \neg A) \lor (\neg A \leftrightarrow C)$$

$$
\begin{array}{lll}
 & (B \to \neg A) \lor (\neg A \leftrightarrow C) & \\
\iff & (\neg B \lor \neg A) \lor (\neg A \leftrightarrow C) & \text{by unraveling } \to \\
\iff & (\neg B \lor \neg A) \lor ((\neg A \to C) \land (C \to \neg A)) & \text{by unraveling } \leftrightarrow \\
\iff & ((\neg B \lor \neg A) \lor (\neg A \to C)) \land ((\neg B \lor \neg A) \lor (C \to \neg A)) & \text{by distributivity} \\
\iff & ((\neg B \lor \neg A) \lor (\neg\neg A \lor C)) \land ((\neg B \lor \neg A) \lor (C \to \neg A)) & \text{by unraveling } \to \\
\iff & ((\neg B \lor \neg A) \lor (A \lor C)) \land ((\neg B \lor \neg A) \lor (C \to \neg A)) & \text{removing double } \neg \\
\iff & ((\neg B \lor \neg A) \lor (A \lor C)) \land ((\neg B \lor \neg A) \lor (\neg C \lor \neg A)) & \text{by unraveling } \to
\end{array}
$$

6: Normal forms for first-order logic and Horn sentences (10 points)

a. Provide a Prenex normal form of the following formula.
You should show all intermediate steps, but you do not have to give justifications.

$$\forall x(\exists y R(y,x) \rightarrow (P(z) \wedge \exists y R(y,x)))$$

b. Provide a Skolem normal form of the following sentence.
You do not have to show all intermediate steps and you do not have to give justifications.

$$\exists w \forall y \exists z \forall u \exists x (Q(x,y,z,u,w) \vee R(y,z))$$

c. Check the satisfiability of the following Horn sentence.

$$B \wedge (\neg A \vee \neg B \vee C) \wedge \neg C \wedge (\neg B \vee A)$$

Use the Horn algorithm and indicate the order in which you assign truth values to the atomic sentences. If you prefer the conditional form, you may also rewrite the formula with $\rightarrow$ and then use the satisfiability algorithm for conditional Horn sentences.

a.
$$\begin{aligned}
& \forall x(\exists y R(y,x) \rightarrow (P(z) \wedge \exists y R(y,x))) \\
\Leftrightarrow\ & \forall x \forall y(R(y,x) \rightarrow (P(z) \wedge \exists y R(y,x))) \\
\Leftrightarrow\ & \forall x \forall y(R(y,x) \rightarrow (P(z) \wedge \exists w R(w,x))) \\
\Leftrightarrow\ & \forall x \forall y(R(y,x) \rightarrow \exists w(P(z) \wedge R(w,x))) \\
\Leftrightarrow\ & \forall x \forall y \exists w(R(y,x) \rightarrow (P(z) \wedge R(w,x)))
\end{aligned}$$

b. $\exists w \forall y \exists z \forall u \exists x (Q(x,y,z,u,w) \vee R(y,z))$

(replace w with a constant c)

$\forall y \exists z \forall u \exists x (Q(x,y,z,u,c) \vee R(y,z))$

(replace z with a unary function $f(y)$)

$\forall y \forall u \exists x (Q(x,y,f(y),u,c) \vee R(y,f(y)))$

(replace x with a binary function $g(u,y)$)

$\forall y \forall u (Q(g(u,y),y,f(y),u,c) \vee R(y,f(y)))$

c. Result:

A	B	C	B	$\wedge$	$(\neg A$	$\vee$	$\neg B$	$\vee$	$C)$	$\wedge$	$\neg C$	$\wedge$	$(\neg B$	$\vee$	$A)$
T	T	T	T		F		F		T		F		F		T

Order of assignments: first (because of the first conjunct) $B :=$ T;
then (because of the fourth conjunct) $A :=$ T;
then (because of the second conjunct) $C :=$ T.

This makes the conjunct $\neg C$ false. The formula is therefore *not* satisfiable.

7: Sets and relations (8 points) Given are the following five sets:

$$A = \{1, 2\}, \;\; B = \{\{1\}, \{2\}\}, \;\; C = \{1, \{1\}\}, \;\; R = \{\langle 1, 2\rangle, \langle 2, 1\rangle\} \;\text{ and }\; S = \{1, \langle 2, 2\rangle\}.$$

For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

a. $C \subseteq A$

b. $\emptyset \in B$

c. $R \subseteq A$

d. $A \cap C \subseteq S$

e. $(A \cap C) \in (B \cap C)$

f. $C \subseteq A \cup B$

g. $A \cap S = \emptyset$

h. R is transitive

a. F

b. F

c. F

d. T

e. T

f. T

g. F

h. F

8: Translating function symbols (7 points) Translate the following sentences using the translation key provided. The domain of discourse is the set of all books and all persons.

$\mathsf{Book}(x)$: x is a book
$\mathsf{Person}(x)$: x is a person
$\mathsf{HasRead}(x, y)$: x has read y

$\mathsf{author}(x)$: the author of x

e: Emily
j: Jane
m: Mansfield Park

a. Emily is a person and Jane is the author of Mansfield Park.

b. Jane has read no books of which Emily is the author.

c. Emily is the author of exactly two books.

a. $\mathsf{Person}(e) \;\wedge\; j = \mathsf{author}(m)$

b. $\neg\exists x(\mathsf{Book}(x) \wedge \mathsf{HasRead}(j, x) \wedge \mathsf{author}(x) = e)$

c. $\exists x \exists y(x \neq y \wedge \mathsf{Book}(x) \wedge \mathsf{Book}(y) \wedge \mathsf{author}(x) = e \wedge \mathsf{author}(y) = e \wedge \forall z((\mathsf{Book}(z) \wedge \mathsf{author}(z) = e) \to (z = x \vee z = y)))$

9: Semantics (10 points)

Let a model $\mathfrak{M}$ with domain $\mathfrak{M}(\forall) = \{1, 2\}$ be given such that

- $\mathfrak{M}(a) = 1$
- $\mathfrak{M}(P) = \{1, 2\}$
- $\mathfrak{M}(R) = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$

Let h be an assignment such that:

- $h(x) = 2$
- $h(y) = 2$

Evaluate the following statements. Follow the truth definition step by step.

a. $\mathfrak{M} \models (R(a, x) \wedge R(a, y)) \to P(a)[h]$

b. $\mathfrak{M} \models \forall x \forall y(P(x) \wedge R(x, y))[h]$

c. $\mathfrak{M} \models \forall x(P(x) \to \exists y(\neg P(y) \vee R(y, y)))[h]$

(a)	Equivalent statements	Reasons for equivalences
1.	$\mathfrak{M} \models (R(a, x) \wedge R(a, y)) \to P(a)[h]$	
	$\Leftrightarrow$	semantics of $\to$
2.	$\mathfrak{M} \not\models R(a, x) \wedge R(a, y)[h]$ or $\mathfrak{M} \models P(a)[h]$	
	$\Leftrightarrow$	semantics of $\wedge$ and De Morgan (meta-level)
3.	$\mathfrak{M} \not\models R(a, x)[h]$ or $\mathfrak{M} \not\models R(a, y)[h]$ or $\mathfrak{M} \models P(a)[h]$	
	$\Leftrightarrow$	sem. of atomic formulas
4.	$\left\langle [\![a]\!]_h^{\mathfrak{M}}, [\![x]\!]_h^{\mathfrak{M}} \right\rangle \notin \mathfrak{M}(R)$ or $\left\langle [\![a]\!]_h^{\mathfrak{M}}, [\![y]\!]_h^{\mathfrak{M}} \right\rangle \notin \mathfrak{M}(R)$ or $[\![a]\!]_h^{\mathfrak{M}} \in \mathfrak{M}(P)$	
	$\Leftrightarrow$	definition of h and $\mathfrak{M}$
5.	$\langle 1, 2\rangle \notin \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$ or $\langle 1, 2\rangle \notin \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$ or $1 \in \{1, 2\}$	

Conclusion: The third disjunct in 5 is true, therefore the statement is true.

(b)	Equivalent statements	Reasons for equivalences
1.	$\mathfrak{M} \models \forall x \forall y(P(x) \wedge R(x, y))[h]$	
	$\Leftrightarrow$	semantics of $\forall$
2.	for all $d \in D$, $\mathfrak{M} \models \forall y(P(x) \wedge R(x, y))[h[x/d]]$	
	$\Leftrightarrow$	semantics of $\forall$
3.	for all $d \in D$, for all $e \in D$, $\mathfrak{M} \models P(x) \wedge R(x, y)[h[x/d][y/e]]$	
	$\Leftrightarrow$	semantics of $\wedge$
4.	for all $d \in D$, for all $e \in D$, $\mathfrak{M} \models P(x)[h[x/d][y/e]]$ and $\mathfrak{M} \models R(x, y)[h[x/d][y/e]]$	
	$\Leftrightarrow$	sem. of atomic formulas
5.	for all $d \in D$, for all $e \in D$, $[\![x]\!]_{h[x/d][y/e]}^{\mathfrak{M}} \in \mathfrak{M}(P)$ and $\left\langle [\![x]\!]_{h[x/d][y/e]}^{\mathfrak{M}}, [\![y]\!]_{h[x/d][y/e]}^{\mathfrak{M}} \right\rangle \in \mathfrak{M}(R)$	
	$\Leftrightarrow$	definition of h and $\mathfrak{M}$
6.	for all $d \in D$, for all $e \in D$, $d \in \{1, 2\}$ and $\langle d, e\rangle \in \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$	

Conclusion: Setting $d = 2$ and $e = 1$ suffices to falsify the second conjunct in 6. The statement is therefore false.

(c) Equivalent statements	Reasons for equivalences
1. $\mathfrak{M} \models \forall x(P(x) \to \exists y(\neg P(y) \lor R(y,y)))[h]$	
$\Leftrightarrow$	semantics of $\forall$
2. for all $d \in D$, $\mathfrak{M} \models P(x) \to \exists y(\neg P(y) \lor R(y,y))[h]$	
$\Leftrightarrow$	semantics of $\to$
3. for all $d \in D$, $\mathfrak{M} \not\models P(x)[h[x/d]]$ or $\mathfrak{M} \models \exists y(\neg P(y) \lor R(y,y))[h[x/d]]$	
$\Leftrightarrow$	semantics of $\exists$
4. for all $d \in D$, $\mathfrak{M} \not\models P(x)[h[x/d]]$ or there exists $e \in D$ s.t. $\mathfrak{M} \models \neg P(y) \lor R(y,y)[h[x/d][y/e]]$	
$\Leftrightarrow$	sem. of $\lor$
5. for all $d \in D$, $\mathfrak{M} \not\models P(x)[h[x/d]]$ or there exists $e \in D$ s.t. $\mathfrak{M} \models \neg P(y)[h[x/d][y/e]]$ or $\mathfrak{M} \models R(y,y)[h[x/d][y/e]]$	
$\Leftrightarrow$	sem. of $\neg$
6. for all $d \in D$, $\mathfrak{M} \not\models P(x)[h[x/d]]$ or there exists $e \in D$ s.t. $\mathfrak{M} \not\models P(y)[h[x/d][y/e]]$ or $\mathfrak{M} \models R(y,y)[h[x/d][y/e]]$	
$\Leftrightarrow$	sem. of atomic formulas
7. for all $d \in D$, $[\![x]\!]^{\mathfrak{M}}_{h[x/d]} \notin \mathfrak{M}(P)$ or there exists $e \in D$ s.t. $[\![y]\!]^{\mathfrak{M}}_{h[x/d][y/e]} \notin \mathfrak{M}(P)$ or $\left\langle [\![y]\!]^{\mathfrak{M}}_{h[x/d][y/e]}, [\![y]\!]^{\mathfrak{M}}_{h[x/d][y/e]} \right\rangle \in \mathfrak{M}(R)$	
$\Leftrightarrow$	definition of h and $\mathfrak{M}$
8. for all $d \in D$, $d \notin \{1,2\}$ or there exists $e \in D$ s.t. $e \notin \{1,2\}$ or $\langle e,e \rangle \in \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle\}$	

Conclusion: For any choice of e we have that $\langle e,e\rangle \in \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle\}$. The second disjunct in 6 is therefore true. It follows that the statement is true.

10: Bonus question (10 points) Give a formal proof of the following formula, in which R is a two-argument predicate symbol and A and B are one-argument predicate symbols. Don't forget to provide justifications. Only use the Introduction and Elimination rules and the Reiteration rule.

$$\vdash (\exists x \exists y R(x,y) \to \forall z B(z)) \lor \exists z (B(z) \to \forall x A(x))$$

1 $\neg[(\exists x \exists y R(x,y) \to \forall z B(z)) \lor \exists z(B(z) \to \forall x A(x))]$

2 $\forall z B(z)$

3 $\exists x \exists y R(x,y)$

4 $\forall z B(z)$ Reit: 2

5 $\exists x \exists y R(x,y) \to \forall z B(z)$ → Intro: 3-4

6 $(\exists x \exists y R(x,y) \to \forall z B(z)) \lor \exists z(B(z) \to \forall x A(x))$ ∨ Intro: 5

7 $\bot$ ⊥ Intro: 6, 1

8 $\neg \forall z B(z)$

9 $\neg \exists z \neg B(z)$

10 [a]

11 $\neg B(a)$

12 $\exists z \neg B(z)$ ∃ Intro: 11

13 $\bot$ ⊥ Intro: 12, 9

14 $\neg\neg B(a)$ ¬ Intro: 11-13

15 $B(a)$ ¬ Elim: 14

16 $\forall z B(z)$ ∀ Intro: 10-15

17 $\bot$ ⊥ Intro: 16, 8

18 $\neg\neg \exists z \neg B(z)$ ¬ Intro: 9-17

19 $\exists z \neg B(z)$ ¬ Elim: 18

20 [c] $\neg B(c)$

21 $B(c)$

22 $\bot$ ⊥ Intro: 21, 20

23 $\forall x A(x)$ ⊥ Elim: 22

24 $B(c) \to \forall x A(x)$ → Intro: 21-23

25 $\exists z (B(z) \to \forall x A(x))$ ∃ Intro: 24

26 $\exists x \exists y (R(x,y) \to \forall z B(z)) \lor \exists z(B(z) \to \forall x A(x))$ ∨ Intro: 25

27 $\bot$ ⊥ Intro: 26, 1

28 $\bot$ ∃ Elim: 19, 20-27

29 $\neg\neg[(\exists x \exists y R(x,y) \to \forall z B(z)) \lor \exists z(B(z) \to \forall x A(x))]$ ¬ Intro: 1-28

30 $(\exists x \exists y R(x,y) \to \forall z B(z)) \lor \exists z(B(z) \to \forall x A(x))$ ¬ Elim: 29